Analytical Functions to Match Size Distributions in Diesel-Sprays

W.Q.Long, H.Hosoya*, T.Mashimo*, K.Kobayashi*, T.Obokata, F.Durstand T.H.Xu****

Department of Mechanical Engineering
Gunma University
1-5-1 Tenjin-cho, Kiryu 376
Japan
* *Unisia Jecs Co.*
** *University of Erlangen-Nürnberg*

ABSTRACT

A general consideration on how to use a continuous analytical probability density function defined in $(-\infty, +\infty)$ to describe a measured droplet size distribution with a limited range and, why the maximum likeness estimation is the best method for parameter estimation in this case is given in detail. A comparative study was carried out for four different size distribution functions commonly used in describing various spray properties based on droplet size, i.e. the log-normal, Rosin-Rammler, Nukiyama-Tanasawa and the log-hyperbolic distribution, in order to investigate the applicability of these functions for Diesel sprays. Data for this study were acquired in three type of Diesel sprays from single-hole, four-holes and conical nozzles under steady and unsteady conditions, using a phase Doppler anemometer. The results of the study, for six sets of experimental data of size distributions show that the log-hyperbolic function presents the best choice.

INTRODUCTION

Improving combustion of diesel engines is one of the most important problems for conservation of the global environment and effective use of fossil resources. Various methods of fuel injection system have been proposed since combustion of diesel engines is affected very much by the form and characteristics of fuel spray and by the injection timing. In the development of spray system, detailed information on droplet size and velocity is relevant to the optimization of the system. In the last ten years, phase Doppler anemometer (PDA) has provided a very suitable technique for measuring local size and velocity distributions in Diesel sprays (1-3). This has resulted in a large amount of data, and the need for a reliable method to extract relevant information from the experiments is obvious. Because the mean diameters are not a sufficient mathematical description of a size distribution, i.e., different distributions may have the same values of mean diameters (4), a suitable analytical function for describing the size distribution in Diesel sprays is required. In this way, variations in the spray character in spatial and in time are reflected by the variations of the parameters in the analytical function. This approach provides an accurate method for correlating the spray character with the operating conditions and for the calculation, simulation and optimization of processes happened in the cylinder, e.g. droplet evaporation.

For these purposes, an analytical function must be accurate enough. Droplet size distributions in Diesel spray have a character: the tails in distribution curves are sometimes extremely short. However, most of the analytical PDF are defined in $(-\infty, +\infty)$ or $(0, +\infty)$. Therefore, how to apply such an analytical function to a measured Diesel spray data is a theoretical problem to be solved in the paper.

Recently, log-hyperbolic distribution functions were applied successfully for many spray flows (4-6) and a comparison of different distribution functions was made for low pressure sprays (6). No similar work has been reported yet, however, to answer which distribution function is most suitable for Diesel sprays and this is the major motivation of the current investigation.

EQUIPMENT AND METHOD EXPERIMENTS

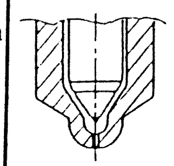
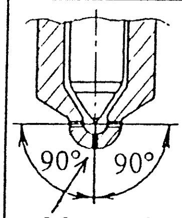
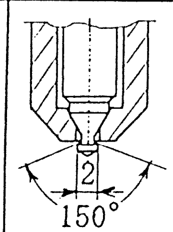
Fuel Injection Nozzle

The details of nozzles A (single hole), B (four holes) and C (conical) used in the experiments are shown in Table 1. Nozzle A has a single-hole on the tip of the nozzle sack. Nozzle B used the same nozzle sack as nozzle A, but three extra injection holes were drilled by electric discharge machining at the 90 degrees positions on the side. For nozzle B, only the central spray was observed, while sprays from other holes were injected into another specially installed sucker. Nozzle C is the same as the pintle nozzle but that a cone of wide angle is attached to the top of the needle. The nozzle exit is 2mm in diameter, and the vertical angle of the cone is 150 degrees. The nozzle is characterized by its capability of spraying particles of uniform and small diameters radially at the ordinary pressure.

Equipment of Experiments

Fig. 1 shows the equipment of experiments (2, 3). In the experiment of unsteady spray, an injection pump was driven by a motor at an injection frequency of 20.8Hz. The pipe connecting the pump and the injection valve was 1.5mm in inside diameter and 600mm long. In the

Table 1 Specifications of three nozzles and injection conditions

Nozzle type	A:Single hole dxl=0.31x 0.6	B: Single hole + 3 holes dxl=0.31x0.6 +dxl=0.36x0.6	C: Conical spray nozzle d=2
Steady injection pressure	9.8 MPa	9.8 MPa	19.6 MPa
Unsteady valve open pressure	29.4 MPa	29.4 MPa	19.6 MPa
Flow rate	19.8 (mm^3 / stroke) (rack 17mm)	37.3 (mm^3/ stroke) (rack 17mm)	12.8 (mm^3 / stroke) (rack 5mm)
Illustration			

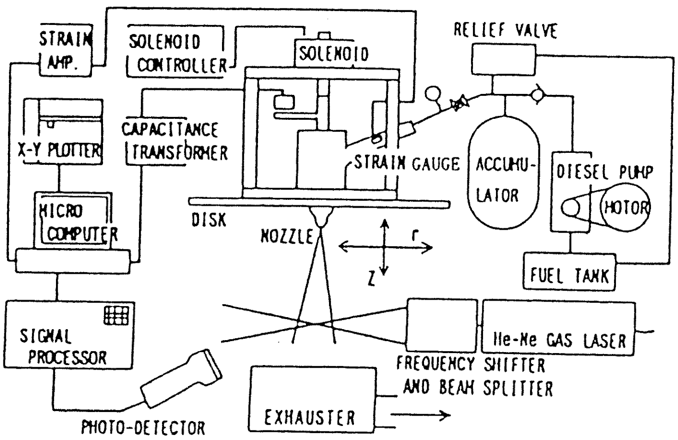


Figure 1: Experimental setup

experiment of steady spray, motor-driven injection pump pressurized fuel and, when the fuel pressure in the accumulator rose to a certain level, a solenoid valve controlled the time of opening the valve. As preliminary experiments disclosed that the flows of spray became steady in 35ms after the beginning of spraying, the duration of injection was set at about 700ms.

Phase Doppler Anemometer

The phase Doppler particle analyzer (PDPA) made by Aerometrics was used to measure the particle diameter and flow velocity of the spray. The diameter of the measuring volume used in this experiment was about 0.7mm. The total intersection angle of incident beams was 2.7 degrees, and forward scattering mode was used with the detector inclined by 30 degrees from the axis of the incident beams.

THEORETICAL CONSIDERATIONS ABOUT TRUNCATED DISTRIBUTIONS

Analytical Function and Truncated Distribution

The sizes of a population of droplets has such a nature: the observed minimum and maximum size are always limited values. The limitation of the tail ends of an observed particle size distribution can be caused by

- a limited measuring range of the instrument, or
- physical reasons of the particle system, for example, the limited energy in the system for producing extremely small droplets, the instability of a large droplet, etc. and can also just be a result of
- limited samples.

To support the last argument, consider just a normally distributed population. For N_{total} samples, a value x_b exists that ensures

$$N_{total} \int_{x_b}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{(x-\mu)^2}{2\sigma^2}] dx \ll 1. \quad (1)$$

So that particle sizes larger than x_b is very possibly not observed in the N_{total} observations.

Since sizes of drops in spray appear to be continuously distributed, a size distribution is described by a continuous analytical probability density function (PDF). Analytical PDFs are usually defined in $(-\infty, \infty)$ for the mathematical generality and simplicity. A measured particle size distribution with limited tail ends x_{min} and x_{max} can be described by a PDF obtained by truncating the tails of an analytical function $f(x, \lambda)$ defined in $(-\infty, \infty)$, where λ is its parameter vector. A reasonable assumption made here is that the population of a particle system is the rest of a population $f(x, \lambda), x \in (-\infty, \infty)$ by taking out all the elements $\zeta < x_{min}$ and $\zeta > x_{max}$. This assumption is true if the truncation is caused by the limited measuring range of the instrument. Based on this concept, it is easy to deduce that the probability of a particle smaller than $x, (x < x_{max})$ is equal to the conditional probability $P(A \mid B)$, where A denotes the event $\zeta < x$ and B the condition $x_{min} < \zeta < x_{max}$. Since

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} , \quad (2)$$

the accumulative particle size distribution can be calculated by

$$G(x) = P(A \mid B) = \frac{\int_{x_{min}}^{x} f(x, \lambda) dx}{\int_{x_{min}}^{x_{max}} f(x, \lambda) dx} . \quad (3)$$

Differential Eq(3) with respect to x, the PDF of the particle size distribution is obtained:

$$g(x) = \frac{1}{\int_{x_{min}}^{x_{max}} f(x, \lambda) dx} f(x, \lambda) . \quad (4)$$

It should be noticed here, that the parameter λ is independent of the tail ends x_{min} and x_{max}. When and only when $x_{min} \to -\infty$ and $x_{max} \to \infty$, $g(x) \to f(x)$. For many particle size distributions, especially drop size dis-

Table 2: Comparison of methods of parameter estimation

	$-8.1 < x < 7.2$					$1.0 < x < 7.2$				
	a	θ	μ	D_{10}	D_{32}	a	θ	μ	D_{10}	D_{32}
true value	0.6000	0.2000	3.0000	2.5232	3.1269	0.6000	0.2000	3.0000	2.7622	3.1726
max. likeness	0.6006	0.1999	2.9997	2.5232	3.1239	0.6007	0.2001	2.9997	2.7621	3.1716
max. likelihood	0.6040	0.1990	2.9986	2.5231	3.1285	0.5060	0.0577	2.8846	2.7490	3.1397
least-squares	0.6060	0.1971	2.9956	2.5240	3.1320	0.5593	0.1719	2.9921	2.5888	3.0312

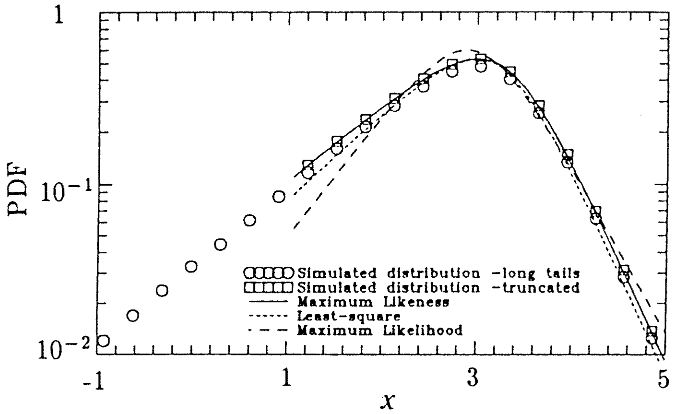


Figure 2: Comparison of methods of parameter estimation

tributions in Diesel sprays, the measured tails are highly truncated. In these cases, the use of $g(x)$, instead of $f(x)$, is recommend for the following reasons:

- Because of the truncation, the value of the empirical PDF is deviated from $f(x, \lambda)$. However, $g(x, \lambda)$ follows the change. This can be best demonstrated by a logarithmic plot (like Fig. 2). The truncation of the tails causes both the empirical PDF and $g(x, \lambda)$ shifted up while the shape of the curves remains unchanged. (refer Eqs.(2)-(4)). Therefore $g(x, \lambda)$ describes the measured distribution better.
- The extremely large or small particles have great contribution to moments, mean diameters, skewness and flatness etc. All these statistic quantities calculated by using $f(x, \lambda)$ may significantly differ from the empirical values.
- The parameter vector λ of the distribution cannot be correctly estimated by the traditional method if $f(x, \lambda)$ is used. This will be discussed in detail in the next subsection.

The analytical function $f(x, \lambda)$ can be used only when the measured size distribution has very long tails, thus $f(x, \lambda)$ and $g(x, \lambda)$ having no practical difference or, in the particle system there exists no physical limitations for the tail ends and the parameter λ has been correctly estimated through $g(x, \lambda)$ or a special estimation procedure.

Strategy to Fit Measured Size Distribution

A measured droplet size distribution is usually not only a highly truncated distribution but also with a considerable uncertainty in the values of its tail ends x_{min} and x_{max}. Especially, when the truncation is dominated by insufficient samples, the more the samples are taken, the longer the tail ends of the distribution would become. The fitting should be done, however, in such a way that the tail ends of the measured distribution have negligible influence on the parameters of the analytical function matched to the measured size distribution.

For estimating the parameters of a theoretical continuous distribution, the maximum likelihood estimation and the method of least squares are commonly used. Such an estimation procedure leads to a parameter estimate $\hat{\lambda}$, with which the analytical function is "most close" to the empirical PDF. Based on the analysis given in the last section, it is evident that for a highly truncated empirical PDF, a truncated analytical function as defined in Eq.(4) yields the correct parameter estimate and this estimate is independent of the tail ends of the distribution. On the other hand, if an analytical function $f(x, \lambda)$ defined in $(-\infty, \infty)$ is used, the estimation procedure will find the "best" estimate, meanwhile the normalization condition

$$\int_{-\infty}^{\infty} f(x, \lambda) dx = 1, \tag{5}$$

which is incorrect in this case, will also be fulfilled. Such an estimate is obviously worse than the one yielded from $g(x, \lambda)$ and is dependent on the extent of truncation.

Barndorff-Nielsen suggested in (7) an approach of parameter estimation for grouped data, as he called, the maximum likeness estimation. In this estimation procedure an estimate of the parameter vector $\hat{\lambda}$ is selected, which maximizes the expression

$$\sum r_i \ln P_i(\hat{\lambda}) \tag{6}$$

where P_i denotes the theoretical probability in the ith group interval,

$$P_i = \frac{\int_{x_{i-1}}^{x_i} f(x, \lambda) dx}{\int_{x_{min}}^{x_{max}} f(x, \lambda) dx} \tag{7}$$

while r_i is its empirical counterparts,

$$r_i = \frac{N_i}{\sum N_i} \tag{8}$$

for a number distribution.

Notice that the maximum likeness estimation is exactly the application of the maximum likelihood principle together with the practical use of the truncated analytical PDF $g(x, \lambda)$ as defined in Eq.(4), it can be deduced immediately that this method should be superior to the maximum likelihood approach and to the method of least squares using $f(x, \lambda)$.

The superiority of the maximum likeness estimation is verified by comparing the mean diameters and moments computed from the fitted function with that directly computed from the data. The conclusion holds also for the

Table 3 Probability Distribution Functions

Log-Normal Distribution	$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}\exp\left[-\frac{1}{2\sigma^2}(\ln x-\mu)^2\right]$ $0\le x<\infty$	(9)
Rosin-Rammler Distribution	$f(x)=\frac{s}{\bar{x}\Gamma(1-3/s)}\left(\frac{x}{\bar{x}}\right)^{s-4}\exp\left[-\left(\frac{x}{\bar{x}}\right)^{s}\right]$ $0\le x<\infty$	(10)
Nukiyama-Tanasawa Distribution	$f(x)=Ax^{\alpha}\exp(-Bx^{\beta})$ $0\le x<\infty$	(11)
Log-Hyperbolic Distribution	$f(x)=A\frac{1}{x}\exp\left[-\alpha\sqrt{\delta^2+(\ln x-\mu)^2}\right]+\beta(\ln x-\mu)$ $-\infty<x<\infty,\ \lvert\beta\rvert<\alpha,\ \delta>0,\ \mu\in(-\infty,\infty)$	(12)
Three-parameter log-hyperbolic (3P-LH) distribution	$f(x)=A\frac{1}{x}\exp\left\{-\frac{a}{a^2\cos^2\theta-\sin^2\theta}\sqrt{(a^2\cos^2\theta-\sin^2\theta)+(\ln x+\mu_0-\mu)^2}-\frac{(a^2+1)\sin\theta\cos\theta}{a^2\cos^2\theta-\sin^2\theta}(\ln x+\mu_0-\mu)\right\}$	(13)

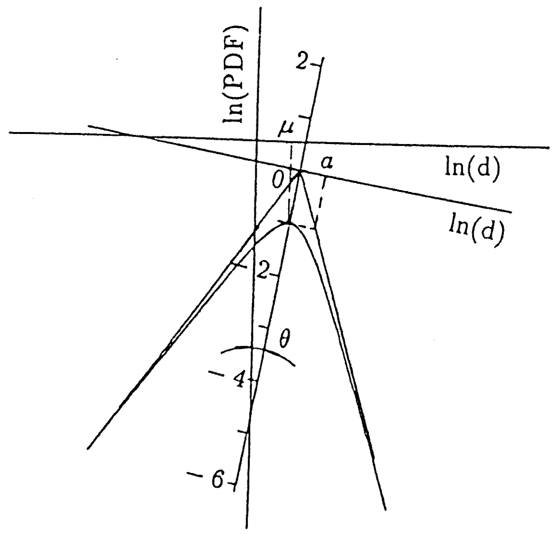


Figure 3: The three parameter log-hyperbolic distribution

agreement of the estimated parameters with the true values in fitting the simulated data, especially in the cases in which the tails of the distribution are incomplete. An example of the comparison of parameter estimation methods is given in Tabal 2 and Fig. 2, where $f(x,\lambda)$ is used. In this investigation 100,000 random number, the values of which were hyperbolically (3P-H) distributed (4), were generated by a computer. The simulated data were than truncated and the 3P-H PDF were fitted with different methods. As shown in the table, changing the degree of truncation of the left tail from $x > -8.1$ to $x > 1.0$ has almost no effect on the maximum likeness estimate. Based on these general considerations, the maximum likeness estimation is used in the current study.

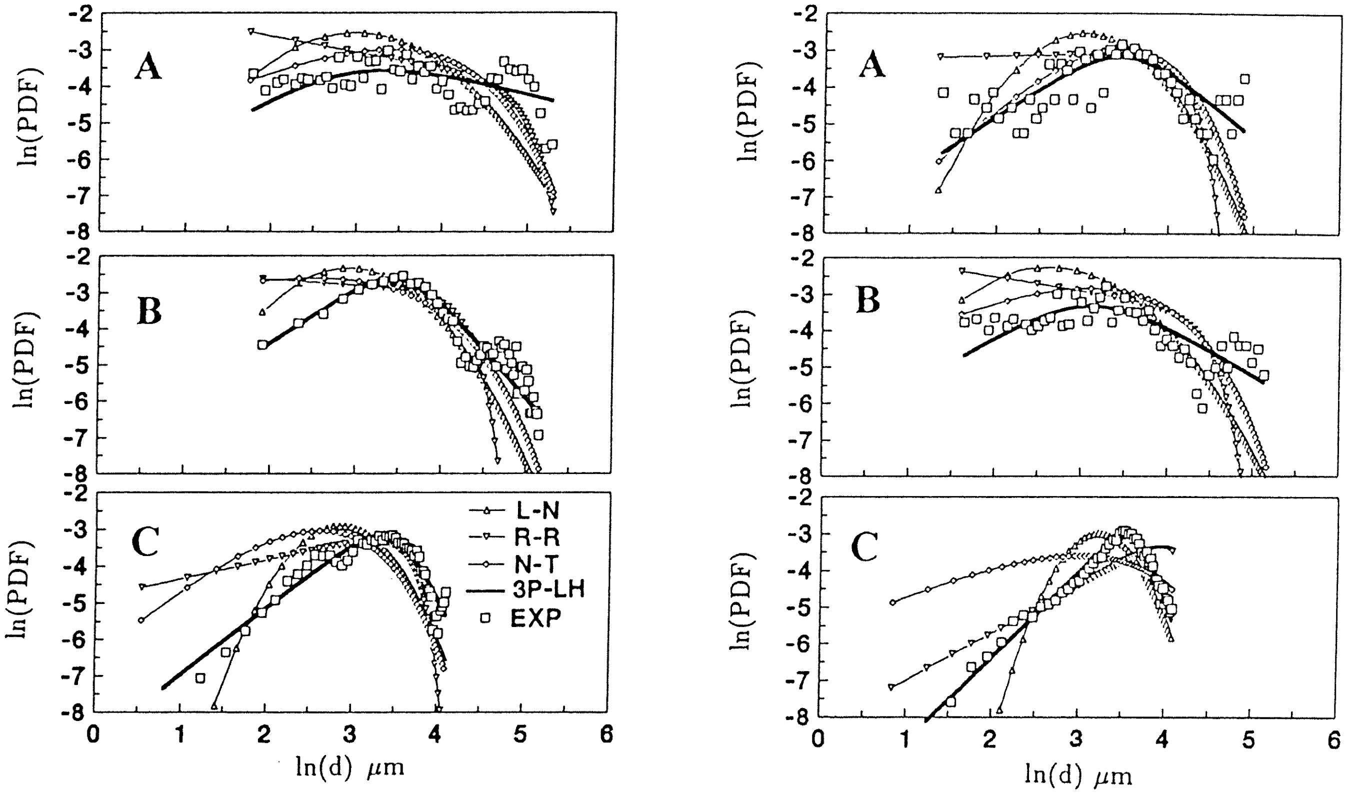
PROBABILITY DISTRIBUTION FUNCTIONS

Four probability distribution functions applied for Diesel type sprays are summarized in Table 3. log-normal (L-N) and Rosin-Rammler (R-R) distributions are well known functions where μ, σ and $\bar{x}$, s are the two parameters respectively characterizing the specified experimental distribution. A power-exponential function proposed by Nukiyama-Tanasawa belongs to the family of the Chi-square distribution where A, B, α and β are the four parameters.

The log-hyperbolic function is a mixture of log-normal distributions where the logarithm of the probability density function is a hyperbola (7). In the current study the three-parameter log-hyperbolic (3P-LH) function is applied to the size distribution of the PDA measurement results. The details of this function are available in references (4). Fig. 3 shows the geometrical meanings of parameter a, θ and μ. a defines the slope of the asymptotes or the opening of the hyperbola, θ is the angle of the axis of the hyperbola relative to the coordinate system, and μ is the location parameter defining the peak of the distribution.

RESULTS

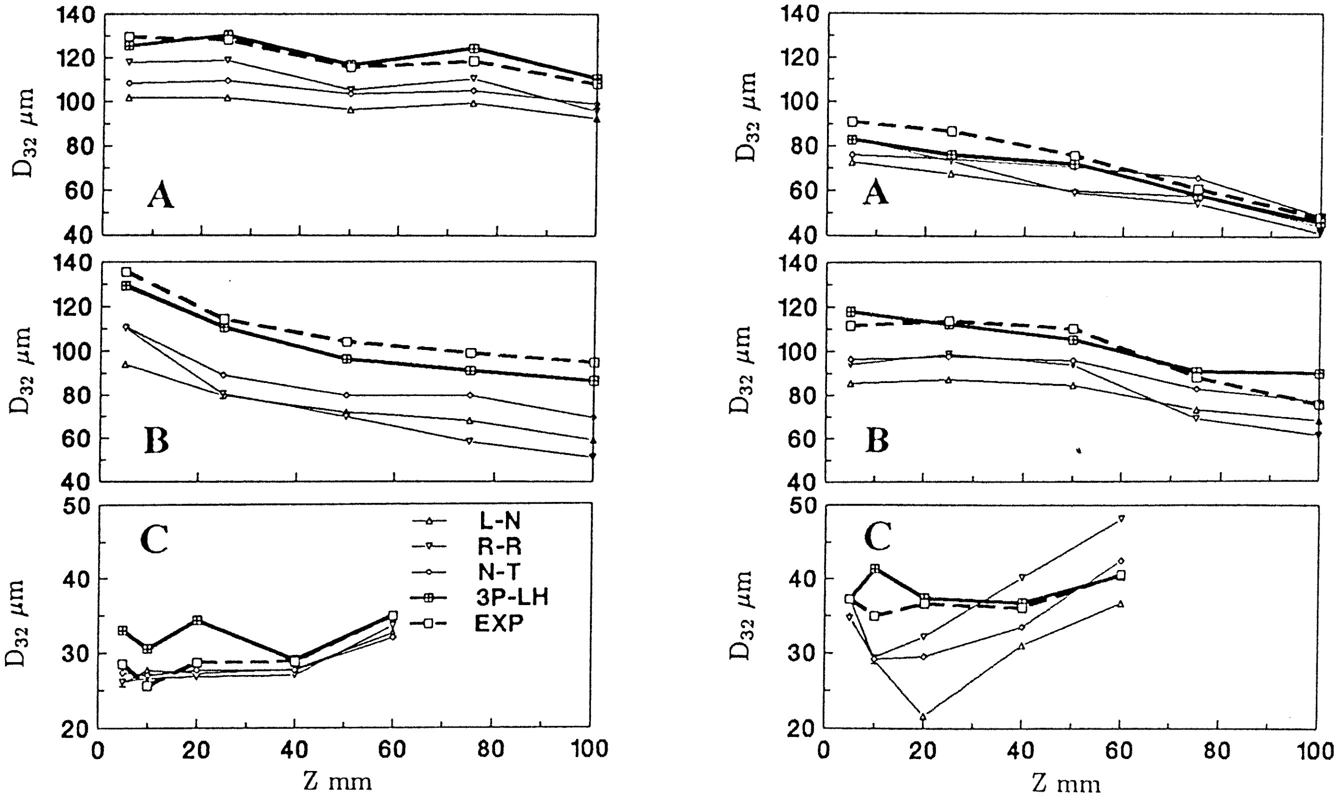
Four size distributions, namely the log-normal (L-N), the Rosin-Rammler (R-R), Nukiyama-Tanasawa (N-T) and the three parameter log-hyperbolic (3P-LH) distribution, were applied to measured data of six types of Diesel sprays. Experimental conditions of three injection nozzles at steady and unsteady conditions are summarized in Table 1. The results of the calculation are shown in Fig. 4 where the experimental data measured at $z = 75$mm or 60mm from the nozzle exit on the center line of the



a) Steady sprays from A, B and C nozzles

b) Unsteady sprays from A, B and C nozzles

Figure 4: Various distributions as applied to Diesel sprays



a) Steady sprays from A, B and C nozzles

b) Unsteady sprays from A, B and C nozzles

Figure 5: Sauter mean diameters along the spray axis

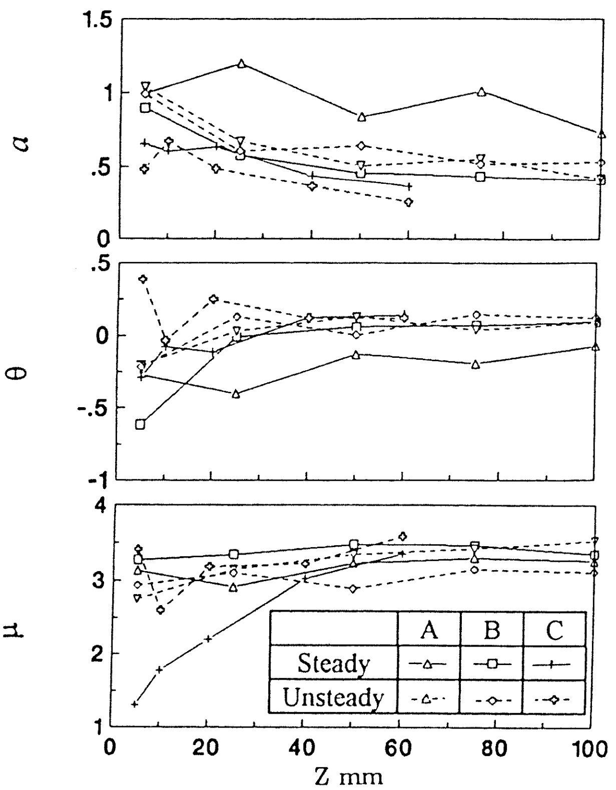


Figure 6: Parameters of 3P-LH function along the injection direction

spray (r=0mm) are plotted by square (□). At the sprays from hole nozzle, it seems that the log-normal distribution does not agree well in the higher size region and the Rosin-Rammler distribution also does not agree well in the lower size region. The Nukiyama-Tanasawa distribution shows better fitting than above functions. At the narrower distributed fine particles of conical spray, poor fittings are observed in the Nukiyama-Tanasawa distribution. The three-parameter log-hyperbolic distribution (3P-LH) shows an excellent fit for all examples.

The values of the Sauter mean diameters D_{32}, as obtained by using the various distributions, are given in Fig. 5. At the hole nozzle sprays and conical spray, decreased and increased mean diameters along the Z axis are shown respectively. The log-hyperbolic distribution provides the best approximation in all cases. Fig. 6 shows the change of the three parameters of a, θ and μ along the axial direction on the center-line of the sprays. The parameters show similar distributions among A, B and C, the exception being nozzle type A in a and θ and nozzle C in μ at steady spray conditions.

CONCLUSIONS

A theoretical study showed that the problem of a truncated distribution can be related to the conditional probability problem thus the PDF of a truncated distribution can be expressed as $g(x,\lambda)$ in Eq.(4), through an analytical PDF $f(x,\lambda)$ defined in $(-\infty,+\infty)$. It has been proved that function $g(x,\lambda)$ describes the measured distribution better than $f(x,\lambda)$ and its parameters can be estimated independent of the tail ends of the measured distribution. The maximum likeness estimation using $f(x,\lambda)$ is equivalent to maximum likelihood estimation using $g(x,\lambda)$ thus providing the best parameter estimation.

To study characteristics of flow and particle diameters in Diesel type sprays, steady and unsteady conical (hollowcone) and hole nozzle (solid) sprays in the air have been analyzed using phase Doppler anemometer (PDA). Four different size distribution functions, i.e. log-normal, Rosin-Rammler, Nukiyama-Tanasawa and three-parameter log-hyperbolic distribution function are compared with experimental data. A comparison of the results calculated for six sets of experimental data for the size distribution shows that the log-hyperbolic function presents the best choice. The variation of spray characteristics along the symmetrical axis of Diesel sprays were presented by means of the variations of the Sauter mean diameter as well as the three parameters of the log-hyperbolic function. The similarities and differences between type of Diesel sprays can be discriminated with the three parameters.

ACKNOWLEDGMENTS

The authors appreciate Mr. Hiroyuki Otsuka and Mr. Tetsuji Koyama, students in Gunma University, for there help in the experiments.

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